Journal of Statistical Physics, Vol. 91, Nos. 1/2, 1998

Anomalies in Multifractal Formalism for Local Time of Brownian Motion

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Received September 29, 1997; final January 6, 1998

The Renyi function for the logical time measure μ of Brownian motion is found. It is shown that this function, the Legendre transform of the multifractal spectrum of μ, and the τ-function derived by the reciprocal measure formalism are not identical. More examples of μ having similar anomalies are discussed.

KEY WORDS: Multifractals; generalized Rényi dimension; fractal Brownian motion; subordinators.

1. INTRODUCTION

Parisi and Frisch[23] introduced the concept of multifractality for probability measures. By definition a measure μ on $J=[0, 1]$ has the multifractal property, if the subsets J_α of points of J having identical local dimensions α (see ref. 21) are fractal. In that case the dimension function $\dim J_\alpha = f(\alpha)$ is the multifractal spectrum of μ. Many examples of multifractal measures (see refs. 3, 6, and 20) have the function $f(\alpha)$ concave in some subinterval of R^1_+. The spectrum can then be found with the help of the so-called multifractal formalism and box counting arguments as follows

$$f(\alpha)=\tau^*(\alpha), \qquad f^*(q)=\tau(q) \tag{1}$$

where $(*)$ is the Legendre transform operation, while τ is a Renyi function of the form

$$\tau_B(q)=\lim_{N\to\infty} \log \Sigma' \mu^q(\Delta_i^{(N)})/\log \Delta_N \qquad |q|<\infty \tag{2}$$

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0022-4715/98/0400-0199$15.00/0

Here, $\gamma = \{\Delta_i^{(N)}\}$ is a partition of the J consisting of equal intervals of size $\Delta_N = 1/N$, the summation involving the terms with $\mu(\Delta_i) \neq 0$. Since (2) uses a partition that involves elements of fixed size, we shall reserve the name of the box τ-function for τ, and accordingly, the notation τ_B. The functional (2) naturally arises in the theory of fully developed turbulence when analyzing the spatial intermittency of dissipation energy.[8]

Proceeding on analogy with the Hausdorff and the packing dimension definitions, Hulsey *et al.*[10] and Olsen[21] put forward alternative definitions of the τ-function. Those definitions are suitable for arbitrary covers γ of the support of μ, and are more natural in view of the multifractal formalism (1). Let $\gamma(\delta) = \{\Delta_i, |\Delta_i| < \delta\}$ be the cover of the support of μ and

$$\Phi(q, \tau) = \sup_{\delta > 0} \inf_{\gamma(\delta)} \Sigma \mu^q(\Delta_i)/|\Delta_i|^\tau \tag{3}$$

where $0^q = \infty$, 1 or 0, when $q < 0$, $=0$, or >0, respectively. The critical value of τ such that

$$\Phi(q, \tau) = \begin{cases} \infty, & \tau > \tau_H(q) \\ 0, & \tau < \tau_H(q) \end{cases}$$

is the desired alternative[21] to (2). It is usually assumed that $\tau_H = \tau_B$ in most cases of interest.[10] This hypothesis is confirmed for a broad class of measures.[21]

Alongside the box counting arguments (1, 2) there is another method of practical interest for calculating $f(\alpha)$. Take two reciprocal measures μ and $\hat{\mu}$ on R_1^+ which means, that the function $\mu([0, x))$ is inverse to $\hat{\mu}([0, x))$ and conversely. Heuristic arguments lead to the following relations between the (τ, f) characteristics of reciprocal multifractal measures:

$$\hat{f}(\alpha) = \alpha f(1/\alpha), \qquad \hat{\tau}(q) = -[-\tau(q)]^- \tag{4}$$

where φ^- is the inverse function to φ. A substantiation of (4) for some class of measures is announced in ref. 24. For example, Cantor's staircase has a two-point multifractal spectrum: $f(\alpha_0) = \alpha_0 = \ln 2/\ln 3$ and $f(\infty) = 1$, while its reciprocal measure has the spectrum[21] exactly as given by (4): $\hat{f}(1/\alpha_0) = 1$ and $\hat{f}(0) = 0$.

This paper is a study of the τ_B-function for the local time measure $L_H(dt)$ of fractional Brownian motion (FBM) with an arbitrary self-similarity index $H \in (0, 1)$. We shall find $\tau_B(q)$ for $q \geqslant 0$ in the general case and for $|q| < \infty$ for Brownian motion ($H = 1/2$). Interest to this problem had arisen in connection with Ya. Sinai's query concerning the multifractal nature of zeroes in Brownian motion;[15] this was discussed on a Sinai–Frisch seminar in 1993. The results reported in refs. 5 and 15 show that the spectrum $f(\alpha)$ is nontrivial for the local time measure of Brownian motion or, more generally, for functions that are reciprocal to stable Levy subordinators. Information on the spectrum $f(\alpha)$ of Levy subordinators themselves can be gathered from ref. 12. Consequently, calculations of τ_B for $L_{1/2}(dt)$ show that $\tau_B \neq \tau_H \neq -[-\hat{\tau}]^-$; that is to say, in particular, the box τ-function does not contain information on the multifractal spectrum of local time measure of Brownian motion. The fact of irregularity in so natural a classical object should be of interest in physical applications where (2) is frequently the basic method[8] for calculating the τ-function which is used to analyze $f(\alpha)$.

The paper is organized as follows. Section 2 contains calculations of the τ_B-function for fractional Brownian motion when $q > 0$; in Section 3 τ_B is calculated completely for Brownian motion; Section 4 is a discussion. The final statements for Sections 2 and 3 are relegated to the Appendix.

2. RENYI FUNCTION FOR FBM LOCAL TIME MEASURE

Let $x_H(t)$, $x_H(0) = 0$ be a continuous centered gaussian process whose structural function is $E\,|x_H(t) - x_H(s)|^2 = |t-s|^{2H}$, $0 < H < 1$, i.e., x_H is fractional Brownian motion. Denote the local time function $x_H(t)$ by $L_H(t)$:

$$L_H(t) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \operatorname{mes}\{s \in (0, t) : |x_H(s)| < \varepsilon\} \tag{5}$$

It is a known fact[9] that L_H can be chosen to be a nondecreasing continuous function.

The process x_H is stochastically self-similar, i.e., $x_H(\lambda t) \stackrel{d}{=} \lambda^H x_H(t)$ where $\stackrel{d}{=}$ stands for equality in the sense of finite dimensional distributions. From this we obtain by using the definition (5) of $L_H(t)$:

$$L_H(\lambda t) \stackrel{d}{=} \lambda^D L_H(t), \qquad D = 1 - H \tag{6}$$

The self-similarity index of L_H also determines the dimension of the topological support of measure $dL_H(t)$, which is in turn identical with the set of zeroes in the process x_H.[13] Let

$$\Sigma_N(q, T) = \sum_{i=0}^{N-1} [L_H(t_{i+1}) - L_H(t_i)]^q \qquad t_i = Ti/N \tag{7}$$

be the function of uniform partition of $(0, T)$. By (6)

$$\Sigma_N(q, T) \stackrel{d}{=} \Delta_N^{qD} \sum_{k=0}^{N-1}{}' [L_H(k+1) - L_H(k)]^q, \qquad \Delta_N = T/N \tag{8}$$

where the prime means that the summation runs through those indices k such that $L_h(k+1) - L_n(k) > 0$. For this reason the study of scaling properties for structural functions (7) of $dL_H(t)$ reduces to the same issue for sums of the type (8)

Theorem 1. (a) When $q \geqslant 0$, one has

$$\lim_{N \to \infty} \frac{\log E\Sigma_N(q, T)}{\log \Delta_N} = D(q-1) \tag{9}$$

(b) When $q \geqslant 0$ and $N = 2^k$, $k = 1, 2,...,$ then

$$\lim_{N \to \infty} \frac{\log \Sigma_N(q, T)}{\log \Delta_N} = D(q-1) \qquad \text{a.s.} \tag{10}$$

Note. In 1993 U. Frisch gave heuristic reasons in favor of (9) for integer $q \geqslant 1$. Special studies of stochastic cascade measures[4, 16] show that limits like (9) and (10) do not necessarily coincide even for q such that $E\Sigma_N(q, T) < \infty$. The belief that space and ensemble averages can be substituted one for another leads to a false interpretation of Kolmogorov's lognormal hypothesis in turbulence.[17]

Theorem 1 is based on two statements to be proved in the Appendix. The first statement is essentially due to Kahane,[13] although not in the present form.

Statement 1. For integer $q \geqslant 1$ and $t \geqslant 0$, the following estimates of the moments $m_q(t) = E\,|L_H(t+1) - L_H(t)|^q$ are true:

$$\tfrac{1}{2} D b_H^q q^{-q/2} p_t < m_q(t) < a_H^q q^{q/2} \Gamma(q+1)\, p_t \tag{11}$$

where $p_t = \min(1, t^{-H})$ and the constants a_H, b_H depend on H only.

Corollary. Where exist constants λ, c_λ and c_H such that

$$P\{L_H(t+1) - L_H(t) > x\} < c_\lambda \exp(-\lambda x^{2/3})\, p_t, \qquad x > 1 \tag{12}$$

$$P\{L_H(t+1) - L_H(t) > 0\} > c_H t^{-H}, \qquad t \geqslant 1 \tag{13}$$

Let us prove this corollary. Relation (12) follows from the Chebyshev inequality in the form $P(\xi > x) < E\varphi(\xi)/\varphi(x)$, where $\xi = L_H(t+1) - L_H(t)$, and $\varphi(x) = \exp(\lambda x^{2/3}) - 1 - \lambda x^{2/3} > 0$, $x > 0$. For indeed, since $\psi(u) = 1 - (1+u)\, e^{-u}$ is an increasing function, it follows that

$$1/\varphi(x) = e^{-\lambda x^{2/3}}/\psi(\lambda x^{2/3}) < e^{-\lambda x^{2/3}}/\psi(\lambda), \qquad x > 1$$

It remains to evaluate

$$E\varphi(\xi) = \sum_{k \geqslant 2} \frac{\lambda^k}{k!} m_{(2/3)k}(t) \tag{14}$$

Because the moment function is logarithmically convex, one has

$$m_{(2/3)k} \leqslant m_{2q}^{1-\varepsilon/3} m_{2q+1}^{\varepsilon/3}, \qquad k = 3q + \varepsilon, \quad \varepsilon = 0, 1 \text{ or } 2$$

From the upper bounds (11) one gets

$$m_{(2/3)k} \leqslant a_H^{(\varepsilon/3)-2} \bar{m}_{2q+2} \tag{15}$$

where $\bar{m}_q$ is the upper estimate for m_q in (11). Combining (11), (14), and (15), one finds that the series (14) converges when $(2\lambda/3)^3\, a_H^2 e < 1$.

All values $\bar{m}_q$, $q \geqslant 1$ are proportional to p_t, hence the same is also true for $E\varphi(\xi)$.

We now prove (13). Let $\mu_q(t)$ denote conditional moments of ξ, given $\xi \neq 0$, and let $p = P\{\xi \neq 0\}$, then $m_q(t) = p\mu_q(t)$. Hence $m_1^2(t)/m_2(t) = p\mu_1^2/\mu_2 \leqslant p$. Substituting in this expression the lower estimate of m_1 and the upper estimate of m_2, one gets the desired estimate of p, see (13). ∎

Statement 2. There exist $t_0 = t_0(H)$ and $c = c(H)$ such that, when $t > t_0$,

$$P(L_H(t+1) - L_H(t) > 0) < c t^{-H} (\ln t)^{1/2} \tag{16}$$

When $t_0^H > 16e$, one may put $c = 30$.

Proof of Theorem 1. From (11) it follows that one has $m_q(t) \asymp t^{-H}$ for integer q, i.e., $c_q^- t^{-H} < m_q(t) < c_q^+ t^{-H}$, $c_q^- > 0$. Since the moment function is logarithmically convex, these estimates still hold for noninteger $q \geqslant 1$. So one gets from (8)

$$E\Sigma_N(q, T) \asymp \Delta_N^{qD} \sum_{t=1}^{N-1} t^{-H} \asymp \Delta_N^{qD} N^{1-H} \tag{17}$$

which yields (9) when $q \geqslant 1$. One can prove (9) for all $q \geqslant 0$ by merely verifying that relation for $q = 0$. From (13) one has

$$E\Sigma'_N(0, T) > c_H \sum_{t=1}^{N-1} t^{-H} \asymp N^D, \qquad D = 1 - H$$

and from (16)

$$E\Sigma'_N(0, T) < c_\varepsilon N^{D+\varepsilon}, \qquad \forall \varepsilon > 0 \tag{18}$$

Since ε is arbitrary, these estimates prove the first statement of Theorem 1.

We now prove (10). A standard application of the Borel–Cantelli lemma to (18) gives $\Sigma'_N(0, T) < N^{D+2\varepsilon}$, $N > N_0(\omega)$ a.s., where N assumes the values in the sequence $N = 2^k$, $k = 1, 2,...$. Hence one gets

$$d = \limsup_{N = 2^k \to \infty} \frac{\log \Sigma'_N(0, T)}{\log N} \leqslant D$$

considering that ε is arbitrary. The value of $\Sigma'_N(0, T)$ determines the number of elements in a uniform partition of $(0, T)$ where the local time increment is different from zero. Therefore $d \geqslant d^-_{\text{box}}$, where d^-_{box} is the lower box dimension of $Z(x_H, T)$, the set of zeroes or x_H in $(0, T)$. However, d^-_{box} is larger than the Hausdorff dimension of $Z(x_H, T)$, which equals D.[13] Consequently, we get

$$\lim_{N = 2^k \to \infty} \log \Sigma'_N(0, T)/\log N = D \qquad \text{a.s.} \tag{19}$$

We are going to show that

$$\lim_{N \to \infty} \log \Sigma_N(1, T)/\log \Delta_N = 0 \qquad \text{a.s.} \tag{20}$$

One has $\Sigma_N(1, T) = L_H(T)$. Consequently, (20) is true, when $p := P(L(T) = 0) = 0$. For, by virtue of (6) the events $A_n = \{L_H(T/n) = 0\}$ have the same probability p. Now event A_1 entails A_n, $n > 1$. Consequently, $A_1 \subset \limsup A_n = A_\infty$. Event A_∞ belongs to the σ-algebra of events generated by $x_H(t)$ in an infinitely small vicinity of $t = 0$. This algebra is trivial (see, e.g., ref. 18 where a canonical representation of $\{x_H(t), t > 0\}$ was derived). It follows that $p(A_\infty) = 0$ or 1. Now one has, $A_1 \subset A_2 ...$ and $p(A_i) = p$. Consequently, $p(A_\infty) = p = 0$ or 1. Since $L(T) \not\equiv 0$ a.s., one has $p = 0$.

Let $\tau_N(q) := \log \Sigma_N(q, T)/\log \Delta_N$. The function $q \to \tau_N(q)$ is concave. Consequently, $\tau_N(q)$, $q \geqslant 1$ $(0 < q < 1)$ lies below (above) the straight line that connects the points $(q_i, \tau_N(q_i))$, $q_1 = 0$ and $q_2 = 1$, i.e., $\tau_N(q) \leqslant \tau_N(0)(1-q) + \tau_N(1)\, q$, $q \geqslant 1$. However, $\lim \tau_N(0) = -D$ a.s. as $N = 2^k \to \infty$, while $\lim \tau_N(1) = 0$ a.s. Consequently, $\limsup_{N=2^k \to \infty} \tau_N(q) \leqslant D(q-1)$, $q \geqslant 1$, and

$$\liminf \tau_N(q) \geqslant D(q-1), \qquad q \in (0, 1) \tag{21}$$

From (17) it follows by the Chebyshev inequality and the Borel–Cantelli lemma that

$$\Sigma_N(q, T) < \Delta_N^{D(q-1)-\varepsilon}, \qquad N = 2^k, \qquad k > k_0(\omega) \qquad \text{a.s.}$$

Hence $\liminf_{N=2^k \to \infty} \tau_N(q) \geqslant D(q-1)$, $q \geqslant 1$. Thus, there exists a.s. $\lim \tau_N(q) = \tau_B(q)$, $q \geqslant 1$, as $N = 2^k \to \infty$, and $\tau_B(q) = D(q-1)$.

Similarly, take $q_1 = 1$ and $q_2 = 2$ in the case $q \in (0, 1)$. Let $l_N(q) = 0$ be the equation of the line that connects the points $(q_i, \tau_N(q_i))$, then $\tau_N(q) \leqslant l_N(q)$, $q \in (0, 1)$. It follows from the above that $l_N(q) \to D(q-1)$ a.s. Hence $\limsup_{N=2^k \to \infty} \tau_N(q) \leqslant D(q-1)$, $0 < q < 1$. The use of (21) will yield the second statement of Theorem 1. ▮

3. RENYI FUNCTION FOR LOCAL TIME IN BROWNIAN MOTION

The process $x_H(t)$, $H = 1/2$ is markovian. This allows one to define τ_B completely. Before we state the relevant result, we refine the estimates (12, 16) in order to be able to judge how far they may be in error in any particular case.

Statement 3. When $H = 1/2$,

$$L_H(t+1) - L_H(t) \overset{\mathrm{d}}{=} (|\eta_1| - \sqrt{t}\, |\eta_2|)_+ \tag{22}$$

where $x_+ = x \cdot \mathbf{1}_{x>0}$ and $\{\eta_i\}$ are standard independent gaussian variables.

Corollaries. (i) One has

$$\begin{aligned} q_t &= P(L_H(t+1) - L_H(t) > 0) \\ &= \frac{2}{\pi} \int_{\sqrt{t}}^{\infty} (1+x^2)^{-1}\, dx \sim \frac{2}{\pi} t^{-1/2}, \qquad t \to \infty \end{aligned} \tag{23}$$

(ii) If $F_t(x)$ is the distribution of $\xi = L_{1/2}(t+1) - L_{1/2}(t)$, then $dF_t(x)/dx = \delta(x)(1-q_t) + f_t(x)$, where

$$f_t(x)/q_t = \Psi\left(\sqrt{\frac{t}{t+1}}\, x\right) \exp\left(-\frac{1}{2}\frac{x^2}{t+1}\right) \cdot \frac{2}{\pi}(1+t)^{-1/2} \tag{24}$$

and

$$\Psi(x) = \int_x^{\infty} \exp(-u^2/2)\, du = \lim_{t\to\infty} f_t(x)$$

Proof. Let $M(a, b) = \max(x_{1/2}(t), t \in (a, b))$. According to Levy,[11] $L_{1/2}(t)$ and $M(0, t)$ are stochastically equivalent processes. Hence

$$\xi = L(t+1) - L(t) \stackrel{d}{=} [M(t, t+1) - M(0, t)]_+ \stackrel{d}{=} [M(0, 1) - M(-t, 0)]_+$$

The last relation incorporates the fact that the increments of $x_{1/2}(t)$ are stationary. Because $x_{1/2}(t)$ is markovian, $M(0, 1)$ and $M(-t, 0)$ are independent. One also has $M(0, a) \stackrel{d}{=} |a|^{1/2} |\eta|$, where η is a standard gaussian variable. That proves (22). Relation (23) immediately follows from (22), because $q_t = P(|\eta_1/\eta_2| > \sqrt{t})$, while η_1/η_2 has a Cauchy distribution. Relation (24) obviously follows from (22). ∎

Let $J_k = [j_k, j_k + 1)$ be consecutive integer intervals in which the Wiener process $w(t) = x_{1/2}(t)$ has zeroes, $j_1 = 0$, while l_k is the increment of local time $L_{1/2}(t)$ in J_k. Suppose $\delta_k + j_k$ is the position of the first zero of $w(t)$ in J_k, $0 \leqslant \delta_k < 1$, $\delta_1 = 0$.

Statement 4. The sequence (l_k, δ_{k+1}) forms a homogeneous Markov chain with a transition probability density $p(l, \delta \mid \delta') = p(l_k = 1, \delta_{k+1} = \delta \mid l_{k-1} = l', \delta_k = \delta')$ which is strictly positive on the entire phase space $[0, \infty) \times [0, 1)$, and

$$p(l, \delta \mid 1-\rho) = \frac{2}{\pi} \sum_{y = k+\delta,\, k \geqslant 0} -\frac{d}{dy}\left[M\left(\sqrt{\frac{y}{\rho+y}}\frac{l}{\sqrt{\rho}}\right)(\rho+y)^{-1/2}\right] e^{-l^2/(2\rho)} \tag{25}$$

where

$$M(u) = e^{u^2/2} \int_u^{\infty} e^{-x^2/2}\, dx$$

while one-dimensional invariant distributions are defined by the densities

$$p_l(x) = \int_x^\infty e^{-u^2/2}\,du = \Psi(x), \qquad x \in [0, \infty) \tag{26}$$

and

$$p_\delta(x) = \tfrac{1}{2}x^{-1/2}, \qquad x \in [0, 1) \tag{27}$$

The proof of Statement 4 is relegated to the Appendix. It applies to all processes $L(t)$ that are reciprocal to stabile Levy processes of index $\alpha \in (0, 1)$.

Theorem 2. For local time measure of Brownian motion there is the a.s. limit

$$\tau_B(q) = \lim_{N=2^k \to \infty} \frac{\log \Sigma_N(q, T)}{\log \Delta_N} = \frac{1}{2}\min(q-1, 2q)$$

Proof. Let $q > -1$. One has $\Sigma_N(q, T) \overset{d}{=} \Delta_N^{q/2} \sum_{k=1}^{\nu_N} l_k^q$, where ν_N is the number of intervals $J_k = [j_k, j_k + 1)$ of a total of $N = T/\Delta$ that contain zeroes of Brownian motion. It follows from Statement 4 that the sequence $\{l_k\}$ can be embedded in a homogeneous Markov chain $\{(l_k, \delta_{k+1})\}$ with a positive transition probability density and a positive stationary density. From this one concludes[19] that, almost surely,

$$\frac{1}{n}\sum_{k=1}^{n} l_k^q \to \int_0^\infty x^q p_l(x)\,dx = \frac{2^{q/2}\Gamma((q/2)+1)}{q+1} = c_q \qquad n \to \infty$$

It was shown above, see (19), that $\nu_T \to \infty$ a.s.

Consequently, $\nu_N^{-1} \sum_{k=1}^{\nu_N} l_k^q \to c_q$ a.s. and so

$$\begin{aligned}\lim_{N=2^k \to \infty} \frac{\log \Sigma_N(q, t)}{\log \Delta_N} &= q/2 + \lim_{N\to\infty} \log \sum_{k=1}^{\nu_N} l_k^q / \log \Delta_N \\ &= q/2 + \lim_{N\to\infty} \frac{\log \nu_N}{\log \Delta_N} \\ &= q/2 + \lim_{N\to\infty} \frac{\log \Sigma_N(0, T)}{\log \Delta_N} \\ &= (q-1)/2\end{aligned} \tag{28}$$

Let $q < -1$ and $\{\tilde{l}_k\}$ be nonzero increments of local time in subintervals of $(0, T)$ of length $1/N$. Using the inequality

$$(\Sigma x_i)^h \leqslant \Sigma x_i^h, \qquad 0 < h < 1, \qquad x_i > 0 \tag{29}$$

one gets $\Sigma \tilde{l}_k^{-|q|} \leqslant (\Sigma \tilde{l}_k^{-|q| h})^{1/h}$. Choose h so that $|q|\, h = 1 - \varepsilon$. From (28) it follows that $\sum_{k=1}^{\nu_N} \tilde{l}_k^{-(1-\varepsilon)} < \Delta_N^{-1}$, $N > N_0(\omega)$.

Hence $\Sigma \tilde{l}_k^{-|q|} \leqslant N^{1/h} = N^{|q|/(1-\varepsilon)}$, $N > N_0(\omega)$. Since ε is arbitrary, we get

$$\limsup_N \frac{\log \Sigma_N(q, T)}{\log 1/\Delta_N} \leqslant |q| \qquad \text{a.s.,} \qquad q < -1$$

We are going to prove the converse inequality by using this author's results.[15] First, we delete in the interval $(0, T)$ all adjoining intervals of the zero set $Z(w(t), T)$ that are longer than Δ_N. The remainder will consist of connected intervals δ_k (Δ-clusters of Z). The points of the lattice $\{i\, \Delta_N\}$ divide the cluster δ_k into intervals $\delta_{k1}, \ldots, \delta_{k\mu(k)}$, $\mu \geqslant 1$. The increments of local time in these intervals are identical with the increments of $L_{1/2}(dt)$ in the corresponding cells $\Delta_i^{(N)} \supset \delta_{kj}$. Thus, $L(\delta_k) = \tilde{l}_{k1} + \cdots + \tilde{l}_{k\mu(k)}$, where $\tilde{l}_{ki} = L(\delta_{ki})$. By (29), $[L(\delta_k)]^{-|q|} < \tilde{l}_{k1}^{-|q|} + \cdots + \tilde{l}_{k\mu(k)}^{-|q|}$, $|q| > 1$, and $\sum_1^{\nu_N} \tilde{l}_k^{-|q|} > \Sigma[L(\delta_k)]^{-|q|}$, $\delta_k \in (0, T)$. From ref. 15 it follows that, for any $\varepsilon > 0$, $\Sigma[L(\delta_k)]^{-|q|} > N^{|q| - \varepsilon}$, $N > N_0(\omega)$, as $N = 2^p \to \infty$. Consequently,

$$\liminf_{N = 2^p \to \infty} \frac{\log \Sigma_N(q, T)}{\log 1/\Delta_N} \geqslant |q|$$

which proves Theorem 2.

4. DISCUSSION

It was shown above that the Renyi function for the local time measure of Brownian motion is

$$\tau_B(q) = \alpha \min(q - 1, 2q) \tag{30}$$

where $\alpha = 1/2$. Indeed, one can assert (see ref. 15 and the proof of Statement 3) that (30) holds for measures $\mathscr{L}_\alpha(dt)$ that are reciprocal to stable Levy processes $H_\alpha(t)$ with index $\alpha \in (0, 1)$. What is the relation between $\tau_B(q)$ and the multifractal spectrum of $\mathscr{L}_\alpha(dt)$? Let $\{\delta_\varepsilon\}$ be the cover of the topological support of $\mathscr{L}_\alpha(dt)$ with ε-clusters, δ_ε. The cover is obtained by eliminating from a line all open intervals of length $\geqslant \varepsilon$ between the points of the support. Following (3), we found[15] such critical values $\tau_{cr}^{(\alpha)}(q)$ of τ that

$$\lim_{\varepsilon \to 0} \sum_{\delta_\varepsilon} L_\alpha^q(\delta_\varepsilon)\, |\delta_\varepsilon|^{-\tau} = \begin{cases} 0, & \tau < \tau_{cr} \\ \infty, & \tau > \tau_{cr} \end{cases} \tag{31}$$

where

$$\tau_{cr}^{(\alpha)}(q) = \alpha \min(q-1, \tfrac{3}{2}q) \tag{32}$$

The ε-clusters for $\mathscr{L}_\alpha$ have lengths ranging between ε and $\varepsilon^{1/\alpha}$ in order of magnitude. However the Renyi functional (2) with $|\Delta| = \varepsilon$ associates all increments $L(\delta_\varepsilon)$ with intervals of length ε. For this reason the use of a functional like (31) to calculate the τ-function becomes essential to describe the multifractal spectrum of $\mathscr{L}_\alpha$. A similar situation must occur for cascade measures with an infinite number of generatrices. For these models see refs. 14 and 24.

Box counting arguments show[15] that the multifractal spectrum of $\mathscr{L}_\alpha(dt)$ is

$$f^{(\alpha)}(h) = \begin{cases} 3\alpha - 2h, & h \in (\alpha, \tfrac{3}{2}\alpha) \\ -\infty, & \text{otherwise} \end{cases} \tag{33}$$

The justification of (33) is supplemented in ref. 5 with the case $\alpha = 1/2$.

It is easy to see that (32) and (33) are consistent, since $f^{(\alpha)*}(q) = \tau_{cr}^{(\alpha)}(q)$. However, $\tau_{cr}(q) \neq \tau_B(q)$. For this reason a formal application of multifractal formalism to τ_B leads to the conclusion that Hölder's exponents for $\mathscr{L}_\alpha(dt)$ are in the interval $(\alpha, 2\alpha)$ rather that in $(\alpha, 1.5\alpha)$. We note that the τ-functions (30) and (32) lose smoothness at different points: $q = -1$ and $q = -2$, respectively. The former ($q = -1$) is critical for the existence of negative moments of $\mathscr{L}_\alpha(\Delta)$. However, because of the strong dependence between $L_\alpha(\delta_\varepsilon)$ and δ_ε, that critical point is considerably shifted to the left when (31) is used.

We now turn to the formalism of reciprocal measures. The multifractal (m.f.) spectrum of paths for H_α, $\alpha \in (0, 2)$ was found in refs. 12 and 22. We note however that the m.f. spectrum of an increasing function and that of the associated measure are generally different. According to ref. 12, $q(t)$ has a smoothness (Hölder's exponent) of order h at t_0, if there exists a constant $c > 0$ and a polynomial p_{t_0} of degree at most $[h]$ such that $|q(t) - p_{t_0}(t)| \leqslant c\,|t - t_0|^h$ in a neighborhood of t_0. By definition, the supremum of such h is a local dimension of $q(t)$ at the point t_0. But, for the measure $dq(t)$, the degree of p_{t_0} is always chosen to be zero. Therefore, the functional m.f.

spectrum of H_α must be identical with the m.f. spectrum of dH_α when $h<1$ and must be not less that this when $h \geqslant 1$.

The paths of H_α, $0<\alpha<1$ are described by a multifractal spectrum[12] of the form

$$F(h)=\begin{cases}\alpha h, & 0<h<1/\alpha \\ -\infty, & h>1/\alpha\end{cases} \tag{34}$$

Hence the m.f. spectrum of dH_α must be $\varphi_\alpha(h)=\alpha h$ for $h \in (0,1)$ and $\varphi_\alpha(h) \leqslant \alpha h$ for $1<h<1/\alpha$. But then, the formal m.f. spectrum of the reciprocal measure is $\hat{\varphi}_\alpha(h):=h\varphi_\alpha(1/h)=\alpha$ for $h>1$ and $\hat{\varphi}_\alpha(h) \leqslant \alpha$ for $\alpha<h<1$. In turn, the Legendre transform φ_α is $\hat{\varphi}^*_\alpha(q)=-\infty$ for $q<0$ and $q-\alpha \leqslant \hat{\varphi}^*_\alpha(q) \leqslant \alpha(q-1)$ for $q>0$. The functions $\hat{\varphi}_\alpha(h)$ and $\hat{\varphi}^*_\alpha(q)$ are sharply at variance with (33) and (32), respectively, indicating that the formalism of reciprocal measures is inapplicable to $\mathscr{L}_\alpha(dt)$.

This circumstance should be borne in mind when describing the solution structure of the inviscid Burgers equation with random initial velocity $v(t)$, refs. 25 and 26. The solution can be described in terms of the convex hull $C(t)$ of the function $y(t)=t^2/2+\int_0^t v(t)\,dt$, $t>0$. It has recently been shown[2] that, when v is a Wiener process, the inverse (reciprocal) function of $C'(t)$, is a Levy process $H(t)$ with the characteristic function

$$\exp(s(\sqrt{2\theta+1}-1))=E\exp\{(H(t+s)-H(t))\,\theta\}$$

The Levy intensity function for jumps in $H(t)$ is $\lambda(x)=(2\pi)^{-1/2}x^{-3/2}\times \exp(-x/2)$, this being different in the exponential factor alone from the same characteristic of the process H_α, $\alpha=1/2$, $\lambda(x)=cx^{-3/2}$. That circumstance does not affect the multifractal properties of Levy processes.[12] Therefore, (34) with $\alpha=1/2$ describes the functional m.f. spectrum of the solution of the Burgers equation for a fixed $t=t_0$. It is of interest to know a m.f. structure of the support of the reciprocal measure $dC(t)$, (see refs. 25 and 26). The support consists of the so-called Lagrange regular points, which give the positions of those fluid particles which have not collided with other particles up to the time $t=t_0$. In view of the above, the m.f. spectrum of these points in terms of $dC(t)$ must be described by (33) with $\alpha=1/2$.

5. APPENDIX: AUXILIARY STATEMENTS

Proof of Statement 1. This proof essentially relies on Kahane's spectral technique[13] developed by him to study local time in fractional Brownian motion.

The local time measure $dL_H(t)$ can be defined in terms of generalized functions:[13]

$$dL_H(t) = \delta(x_H(t))\,dt = \frac{1}{2\pi}\int e^{ix_H(t)\lambda}\,d\lambda\,dt$$

For this reason one has for integer q

$$\begin{aligned} m_q(t) &= E[L(t+1) - L(t)]^q \\ &= (2\pi)^{-q}\,E\int_t^{t+1} d^q s \int_{-\infty}^{\infty} d^q\lambda \exp\left(i\sum_{i=1}^{q} x_H(s_i)\,\lambda_i\right) \end{aligned} \tag{A1}$$

where $d^q x = dx_1,..., dx_q$. Since $x_H(t)$ is gaussian,

$$E\exp\left(i\sum_1^q x_H(s_j)\,\lambda_j\right) = \exp(-\tfrac{1}{2}\Psi(\lambda, s)) \tag{A2}$$

where

$$\Psi(\lambda, s) = E\left|\sum_{j=1}^{q} \lambda_j x_H(s_j)\right|^2$$

Let us use the spectral representation of $x_H(t)$[13] in terms of complex-valued white noise $Z'(y)$:

$$x_H(t) = d_H^{-1}\int (e^{ity} - 1)\,|y|^{-(1/2)-H}\,dZ(y)$$

where $d_H^2 = -4\Gamma(-2H)\cos\pi H$. Hence

$$\Psi(\lambda, s) = d_H^{-2}\int\left|\sum_{j=0}^{q} \lambda_j e^{is_j y}\right|^2 |y|^{-(1+2H)}\,dy \tag{A3}$$

where $\lambda_0 = -\sum_1^q \lambda_j$, $s_0 = 0$. Combining (A1, A2, A3), one gets

$$m_q(t) = (2\pi)^{-q}\int_t^{t+1} d^q s \int_{-\infty}^{\infty} d^q\lambda \exp(-\tfrac{1}{2}\Psi(\lambda, s)) \tag{A4}$$

Estimate of m_q from Above. Let φ be a smooth finite function $\varphi \in C^\infty(|x| < 1)$, $\varphi \geqslant 0$, $\varphi(0) = 1$ and $\varphi(t) = \int e^{iut}\rho(u)\,du$.

The use of Schwartz' inequality gives

$$\left|\varepsilon^{-1}\sum_{j=0}^{q}\varphi((s_j-s)/\varepsilon)\,\lambda_j\right|^2$$

$$=\left|\int_{-\infty}^{\infty}\Sigma\lambda_j e^{iu(s_j-s)}q(\varepsilon u)\,du\right|^2$$

$$<\int_{-\infty}^{\infty}\left|\sum_{j=0}^{q}\lambda_j e^{iu(s_j-s)}\right|^2|u|^{-1-2H}\,du\int|\rho^2(\varepsilon u)|\,|u|^{1+2H}\,du$$

$$=c_H\varepsilon^{-2-2H}\Psi(\lambda,s)$$

where

$$c_H=d_H^2\int|\rho(u)|^2\,|u|^{1+2H}\,du \tag{A5}$$

i.e.,

$$\Psi(\lambda,s)\geqslant\varepsilon^{2H}\left|\sum_{j=0}^{q}\varphi((s_j-s)\,\varepsilon^{-1})\,\lambda_j\right|^2c_H^{-1} \tag{A6}$$

Let $t<s_1\cdots<s_p<t+1$. Put in (A6), successively, $s=s_k$, $\varepsilon=s_{k+1}-s_k$, $k=1,2,\ldots$. Then, since φ is finite, one gets a series of inequalities:

$$\Psi(\lambda,s)>c_H^{-1}\,|s_{k+1}-s_k|^{2H}\left|\sum_{j=0}^{k-1}\varphi\left(\frac{s_j-s_k}{s_{k+1}-s_k}\right)\lambda_j+\lambda_k\right|^2,\qquad k=1,\ldots,q-1 \tag{A7}$$

When $t\geqslant 1$, the constant in front of λ_0 vanishes, because $s_0=0$, $s_k>t>1$, while $s_{k+1}-s_k<1$. The series of inequalities (A7) will then be supplemented with (A6) where $s=0$ and $\varepsilon=t$:

$$\Psi(\lambda,s)>c_H^{-1}t^{2H}\left|\sum_{1}^{q-1}\lambda_j+\lambda_q\right|^2,\qquad t>1$$

When $t\leqslant 1$, (A7) will be supplemented with (A6) where $\varepsilon=s_1$ and $s=0$, i.e.,

$$\Psi(\lambda,s)>c_H^{-1}s_1^{2H}\,|\lambda_0|^2,\qquad t<1$$

Add all estimates of Ψ, q in number. When $t>1$, one gets

$$\Psi(\lambda, s) \geqslant c_H^{-1} q^{-1} [(s_2 - s_1)^{2H} \lambda_1^{*2} + \cdots + (s_q - s_{q-1})^{2H} \lambda_{q-1}^{*2} + t^{2H} \lambda_q^{*2}] =: \Psi^*$$

where new variables have been introduced:

$$\lambda_k^* = \sum_{j=l}^{k-1} \varphi\left(\frac{s_j - s_k}{s_{k+1} - s_k}\right) \lambda_j + \lambda_k, \qquad k = 1,\dots, q-1$$

$\lambda_q^* = \sum_1^{q-1} \lambda_i + \lambda_q$, and $l=1$.

When $t<1$, one has $l=0$, and the terms $t^{2H}\lambda_q^2$ will be replaced with $s_1^{2H}\lambda_0^*$, $\lambda_0^* = \lambda_0$. Let us integrate the inequality

$$\exp(-\tfrac{1}{2}\Psi(\lambda, s)) \leqslant \exp(-\tfrac{1}{2}\Psi^*(\lambda, s))$$

over $\lambda = (\lambda_1,\dots, \lambda_q)$ for $s = (s_1,\dots, s_q)$ in the cone $K_\pi = \{s: t < s_1 < \cdots < s_q < t+1\}$. New variables, λ^*, will be used when integrating the right-hand side. Then one gets for $t>1$

$$\int \exp(-\tfrac{1}{2}\Psi(\lambda, s))\, d^q\lambda < (2\pi c_H q)^{q/2} \prod_{j=1}^{q-1} (s_{j+1} - s_j)^{-H} t^{-H} \tag{A8}$$

When $t<1$, the factor t^{-H} will be replaced with s_1^{-H}. Integration of (A8) over the cone K_π yields

$$\begin{aligned}\int_{K_\pi} d^q s \int \exp(-\tfrac{1}{2}\Psi(\lambda, s))\, d^q\lambda &< (2\pi c_H q)^{q/2} \left[\int_0^1 u^{-H}\, du\right]^{q-1} t^{-H} \\ &\leqslant (2\pi c_H q)^{q/2} D^{-q} t^{-H}\end{aligned} \tag{A9}$$

where $D = 1-H$. When $t<1$, the factor t^{-H} is replaced with 1. The cube $[t, t+1]^q$ can be divided into $q!$ cones K_π. Therefore (A4) combined with (A9) give the right part of (11) with $a_H = (c_H/(2\pi))^{1/2}(1-H)^{-1}$ and c_H in (A5).

Estimate of $m_q(t)$ from Below. One has

$$\begin{aligned}\Psi(s, \lambda) &= E\left|\sum_{j=1}^{q} \lambda_j x_H(s_j)\right|^2 \\ &= E\left|\sum_{j=2}^{q} \lambda_j (x_H(s_j) - x_H(s_1)) + \lambda_1^* x_H(s_1)\right|^2\end{aligned}$$

where $\lambda_1^* = \sum_1^q \lambda_j$. The use of Cauchy's inequality gives

$$\Psi(s, \lambda) \leqslant q \left(\sum_2^q |\lambda_j| \, E(x_H(s_j) - x_H(s_1))^2 + |\lambda_1^*|^2 \, E\, |x_H(s_1)|^2 \right)$$

$$\leqslant q \left(\sum_{j=2}^q \lambda_j^2 \, |s_j - s_1|^{2H} + |\lambda_1^*|^2 \, |s_1|^{2H} \right)$$

Hence

$$\int \exp\left(-\frac{1}{2} \Psi(s, \lambda) \right) d^q\lambda \geqslant \left(\frac{2\pi}{q} \right)^{q/2} \prod_{j=2}^q (s_j - s_1)^{-H} \, |s_1|^{-H}$$

and $m_q(t) \geqslant (2\pi)^{-q} \, (2\pi/q)^{q/2} \, I$, where

$$I = \int_0^1 \left(\int_0^1 \frac{dx}{|x+\theta|^H} \right)^{q-1} \frac{d\theta}{(t+\theta)^H} > D^{1-q} \int_0^1 \frac{d\theta}{(t+\theta)^H}$$

$$> D^{1-q}(1+t)^{-H} > D^{1-q} 2^{-H} \min(1, t^{-H})$$

Finally, one gets the left part of (11) with $b_H = (2\pi)^{-1/2} \, (1-H)^{-1}$.

Proof of Statement 2. Let us estimate $q_t := P(L(t+1) - L(t) > 0)$ from above.

By (6), $L(t+1) - L(t) \overset{d}{=} [L(1 + 1/t) - L(1)] \, t^D$. Hence

$$q_t \leqslant P(\exists t \in \Delta = (1, 1 + t^{-1}) : x_H(t) = 0)$$

$$\leqslant \int_\delta^\infty 2P(x_H(1) \in da) \, P_{-a}(\max_\Delta \, (x_H(t) - x(1)) > a) + P(|x(1)| < \delta)$$

where P_{-a} is the conditional measure of $x_H(t)$ given $x_H(1) = -a$. Since $x_H(t)$ is gaussian, one has

$$x_H(s) - x_H(1) = y(s) + (b_H(s, 1) - 1) \, x_H(1) \qquad \text{(A10)}$$

where $b(s, s') = E x_H(s) \, x_H(s')$, and $y(s)$ is a centered gaussian process that is independent of $x_H(1)$, and

$$E \, |y(s) - y(s')|^2 = |s - s'|^{2H} - [b_H(s, 1) - b_H(s', 1)]^2 \qquad \text{(A11)}$$

Using (A10) and the requirement $x_H(1) = a$, one gets

$$P_{-a}(\max_{s \in \Delta} \, (x_H(s) - x_H(1)) > a) < P(\max_{s \in \Delta} \, y(s) > a(1 - \rho))$$

where $\rho = \max_{s \in \Delta}(b_H(s, 1) - 1) = \max_{s \in \Delta}(|s|^{2H} - 1 - (s-1)^{2H})/2$ i.e., $\rho < 1/2$ for $t > 1$. Consequently,

$$q_t \leqslant P(\max_{s \in \Delta} y(s) > \tfrac{1}{2}\delta) + P(|x_H(1)| < \delta) \tag{A12}$$

where the second term on the right is obviously less that δ. Fernique's estimate[7] will be used for the first term in (A12):

$$P\left(\max_{\Delta} y(s) > x\left[\sigma_\Delta + c\int_1^\infty \varphi_\Delta(2^{-u^2})\,du\right]\right) \leqslant 10\int_x^\infty e^{-u^2/2}\,du \tag{A13}$$

where $x > \sqrt{1 + 4\log^2}$, $c = 2 + \sqrt{2}$, $\sigma_\Delta^2 = \max_{s \in \Delta} Ey^2(s)$ and

$$\varphi_\Delta^2(h) = \max(E[y(s) - y(s')]^2;\, s, s' \in \Delta,\, |s - s'| < h\,|\Delta|)$$

From (A10) one has $\sigma_\Delta^2 < \max_{s \in \Delta} E\,|x_H(s) - x_H(1)|^2 = |\Delta|^{2H}$ and, from (A11),

$$\varphi_\Delta^2(h) < \max(|s - s'|^{2H} : s, s' \in \Delta,\, |s - s'| < h\Delta) = |\Delta|^{2H}\,h^{2H}$$

Hence (A13) yields

$$P(\max_{s \in \Delta} y(s) > x\,|\Delta|^H\,c(H)) \leqslant 10\int_x^\infty e^{-u^2/2}\,du$$

where

$$\begin{aligned} c(H) &= 1 + (2 + \sqrt{2})\int_1^\infty e^{-u^2 H \ln 2}\,du \\ &\leqslant 1 + (2 + \sqrt{2})(\pi/2)^{1/2}/(2H\ln 2)^{1/2} \\ &\leqslant 15\sqrt{2H} \end{aligned}$$

Let x and δ be such that $A := x\,|\Delta|^H\,c(H) < \delta$, and $B := 10\int_x^\infty e^{-u^2/2} < \delta$. Then one gets from (A12) and (A13) the result $q_t \leqslant 2\delta$. The above inequalities can be satisfied by setting $x = |2H\ln|\Delta||^{1/2}$ and $\delta = 15\,|\Delta|^H \times |\ln|\Delta||^{1/2}$. For indeed, when $t = |\Delta|^{-1} \geqslant 2$, one has

$$\begin{aligned} A &< (2H\ln|\Delta|^{-1})^{-1/2}\,|\Delta|^H \cdot 15/(2H)^{1/2} = \delta \\ B &< 10\exp(-x^2/2) \cdot 2/(\sqrt{x^2 + 2} + x) \leqslant 10\sqrt{2}\,|\Delta|^H < \delta \end{aligned}$$

The estimate of B is based on Komatsu's inequality:[11]

$$e^{x^2/2}\int_x^{\infty} e^{-y^2/2}\,dy \leqslant 2/(\sqrt{2+x^2}+x)$$

Fernique's estimate is true when $x = |2H \ln |\Delta||^{1/2} > \sqrt{5}$, i.e., when $t^{2H} > 5$. Thus, $q_t \leqslant 2\delta = 30t^{-H}\sqrt{\ln t}$ when $t^{2H} > 5$. The statement is proven.

Proof of Statement 4. *Markov Property of the Sequence* (l_k, δ_{k+1}). The function $H(x)$, which is continuous on the right and is reciprocal to the local time function $L_{1/2}(t)$, is a stable Levy process of index $\alpha = 1/2$, ref. 11. The process $H(x)$ has independent increments and possesses the strict Markov property, i.e., the process $H^*(x) = H(\tau + x) - H(\tau)$ has the same probability structure as $H(x)$ for any stopping time τ and is independent of $\{H(x), x < \tau\}$, ref. 1.

Suppose $\delta_1 = 0$, $H_1(x) = H(x)$. Let us define recurrently the quantities $\{\tau_i, \delta_i, H_i(x)\}$, where the function H_i is stochastically equivalent to $\{H(x), x \geqslant 0\}$. One has

$$\tau_i = \inf\{x: \delta_i + H_i(x) > 1\}, \qquad i = 1, 2,...$$

$$H_{i+1}(x) = H_i(\tau_i + x) - H_i(\tau_i)$$

$$\delta_{i+1} = \text{fractional part of } (\delta_i + H_i(\tau_i) - 1)$$

The quantity τ_i determines the stopping time for the (continuous on the right) process $H_i(x)$. For this reason, if $\{H_i(x), x > 0\} \stackrel{d}{=} \{H(x), x > 0\}$, then $\{H_{i+1}(x), x > 0\} \stackrel{d}{=} \{H(x), x > 0\}$ and H_{i+1} is independent of $\{\tau_i, \delta_{i+1}\}$. The distribution of $\{\tau_i, \delta_{i+1}\}$ is completely specified by the quantity δ_i and the process $\{H_i(x), x > 0\} \stackrel{d}{=} \{H(x), x > 0\}$. For this reason the sequence $\{\tau_i, \delta_{i+1}\}$ is a Markov chain. It is easy to see that, if one denotes by $J_k = [j_k, j_k + 1)$ successive intervals of an integer lattice where $L_{1/2}(J_k) = l_k \neq 0$, then $\tau_k = l_k$ and $\delta_{k+1} + j_k$ is the first zero of $x_{1/2}(t)$ in J_{k+1}.

The Distribution of (l_k, δ_{k+1}). Let $H_\alpha(x)$ be a stable Levy process of index $\alpha \in (0, 1)$ that is continuous on the right, $\tau(h) = \inf\{h \geqslant 0, H_\alpha(x) > h\}$ is the time of the first exceedance of level h, and $K(h) = H_\alpha(\tau(h)) - h$ is the exceedance itself. When $\alpha = 1/2$, then $H_{1/2}(x) = H(x)$ and the distribution of $(\tau(a), \{K(a)\})$, where $0 \leqslant \{K\} < 1$ is the fractional part of K, is identical with the conditional distribution of (l_k, δ_{k+1}) given $\delta_k = 1 - a$, $a \in (0, 1]$. It is therefore sufficient to find the distribution of $(\tau(a), K(a))$.

For an arbitrary subordinator $\xi(t)$ (a Levy process with range of values $[0, \infty)$) one has[1]

$$I := \int_0^\infty e^{-qa} E e^{-x\tau(a) - yK(a)}\, da = \frac{K(x, q) - K(x, y)}{(q - y)\, K(x, q)}$$

where

$$K(x, y) = \exp \int_0^\infty \frac{dt}{t} E(e^{-t} - e^{-xt - y\xi(t)})$$

When $\xi(t) = H_\alpha(t)$, the distribution of $H_\alpha(t)$ has the density

$$p_H^{(\alpha)}(x \mid t) = p_H^{(\alpha)}(xt^{-1/\alpha} \mid 1)\, t^{-1/\alpha} > 0, \qquad x > 0 \tag{A14}$$

and the Laplace transform is

$$E e^{-yH_\alpha(t)} = \exp(-t(cy)^\alpha) \tag{A15}$$

where c is a normalizing constant. Hence $K(x, y) = x + (yc)^\alpha$ and

$$I := \frac{q^\alpha - y^\alpha}{q - y} \frac{c^\alpha}{(cq)^\alpha + x} \tag{A16}$$

Recall that

$$\frac{q^\alpha - y^\alpha}{q - y} = \int_0^\infty \int_0^\infty e^{-qa - yk} \frac{da\, dk}{(a + k)^{\alpha + 1}} \frac{1}{|\Gamma(-\alpha)|} \tag{A17}$$

and

$$\frac{c^\alpha}{(cq)^\alpha + x} = c^\alpha \int_0^\infty \int_0^\infty e^{-qa - xt} p_t^{(\alpha)}(a)\, da\, dt \tag{A18}$$

The last relation obviously follows from (A15). Combining (A16), (A17), and (A18), one gets

$$P(\tau(a) \in dt, K(a) \in dk) = c^\alpha \frac{(a + k)^{-\alpha - 1}}{|\Gamma(-\alpha)|} * p_H^{(\alpha)}(a \mid t) \tag{A19}$$

where the convolution is over the parameter a. From this one finds the conditional density $p(l_k, \delta_{k+1} \mid \delta_k = 1 - a)$ for the process $L_\alpha(t)$ that is reciprocal to $H^{(\alpha)}$:

$$p(t, \delta \mid 1 - a) = \sum_{n=0}^\infty c^\alpha \frac{(a + n + \delta)^{-\alpha - 1}}{|\Gamma(-\alpha)|} * p_H^{(\alpha)}(a \mid t)$$

When $\alpha = 1/2$, the distribution $p_H^{(\alpha)}(a \mid t)$ can be found in explicit form:[11] $(2\pi)^{-1/2}\, ta^{-3/2} \exp(-t^2/(2a))$ where the normalizing constant is $c = 2$. Therefore, we find an explicit form for (A19), when $\alpha = 1/2$:

$$P(\tau(a) > t,\, K(a) > k) = \frac{2}{\pi} \int_{\sqrt{k/a}}^{\infty} \exp\left(-\frac{u^2+1}{2}\frac{t^2}{a+k}\right)\frac{du}{u^2+1}$$

The last expression yields (25).

In the general case $\alpha \in (0, 1)$, one finds from (A19) and (A14) the one-dimensional distribution of $K(a)$:

$$P(K(a) \in dk) = \frac{\sin \pi\alpha}{\pi} \frac{dk}{(k+a)(k/a)^{\alpha}} \tag{A20}$$

The Invariant Distribution of (l_k, δ_{k+1}). The states of the chain (l_k, δ_{k+1}) are only governed by the state of the second coordinate at the preceding step. For this reason it is sufficient to find the invariant distribution of δ_k. We show that, when $\alpha \in (0, 1)$, the measure $d\sigma(\delta) = (1-\alpha)\, \delta^{-\alpha}\, d\beta$, $0 \leqslant \delta < 1$ is invariant for δ_k. In view of (A20), it is required to verify

$$\sum_{n=o}^{\infty} \frac{\sin \pi\alpha}{\pi} \int_0^1 (n+\delta+1-a)^{-1}\, (n+\delta)^{-\alpha}\, (1-a)^{\alpha}\, d\sigma(a) = \sigma'(\delta)$$

Expanding $(n+\delta+1-a)^{-1}$ in powers of $1-a$ and integrating this, one obtains for the left-hand side: $\sum_{n=0}^{\infty} [(n+\delta)^{-\alpha} - (n+\delta+1)^{-\alpha}](1-\alpha)$ which is identical with $\sigma'(\delta)$. By (A19) the conditional density of l_k given $\delta_k = 1-a$ is

$$c^{\alpha} \frac{a^{-\alpha}}{\Gamma(1-\alpha)} * p_H^{(\alpha)}(a \mid t)$$

Hence the invariant distribution of l_k is

$$\begin{aligned} p_l^{(\alpha)}(t) &= (1-\alpha)\, a^{-\alpha} * c^{\alpha} \frac{a^{-\alpha}}{\Gamma(1-\alpha)} * p_H^{(\alpha)}(a \mid t)\Big|_{a=1} \\ &= c^{\alpha}\Gamma(2-\alpha) \int_0^1 \frac{(1-a)^{1-2\alpha}}{\Gamma(2-2\alpha)}\, p_H^{(\alpha)}(a \mid t)\, da \end{aligned}$$

In view of (A14), one has $p_l^{(\alpha)}(t) = c^{\alpha}\Gamma(2-\alpha)\, \varphi(t^{-1/\alpha})\, t^{1-2\alpha/\alpha}$, where

$$\varphi(x) = \int_0^x \frac{(x-u)^{1-2\alpha}}{\Gamma(2-2\alpha)}\, p_H^{(\alpha)}(u \mid 1)\, du$$

When $\alpha = 1/2$, one obviously has (26). The Laplace transform of φ is $\lambda^{2\alpha-2}e^{-(\lambda c)^{\alpha}}$, and so $\varphi(t) \sim t^{1-2\alpha}/\Gamma(2-2\alpha)$, $t \to \infty$, or $p_l^{(\alpha)}(t) = c^{\alpha}\Gamma(2-\alpha)/\Gamma(2-2\alpha) \cdot (1+o(1))$, $t \to 0$. Therefore $\int t^q p_l^{(\alpha)}(t)\, dt < \infty$, iff $q > -1$.

ACKNOWLEDGMENTS

I am grateful to U. Frisch for helpful discussions and for the hospitality at the Observatoire de la Cote d'Azur where this work was carried out. I also thank anonymous reviewers for careful reading of the manuscript and useful comments.

The work was supported by the French Ministry of Higher Education and in part by the Russian Foundation for Basis Research (Grant 96-01-0037) and by the National Science Foundation (Grant EAR-9423818)

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